Domain Theory and Nonmonotonic Reasoning

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Coherent algebraic cpos

cpo: directed complete partial order with bottom $(D, \sqsubseteq)$

$c \in \mathsf{K}(D)$ (compact) iff $(\forall A \text{ directed})(d \sqsubseteq \bigsqcup A \implies (\exists a \in A) d \sqsubseteq a)$

cpo *algebraic*: $(\forall x)\,(x = \bigsqcup(x{\downarrow} \cap \mathsf{K}(D)))$

Scott topology: base $\{\uparrow c \mid c \in \mathsf{K}(D)\}$

coherent: finite intersections of compact-opens are compact-open

Examples: Finite posets with bottom. Powersets. $\mathbb{T}^\omega$.

Smyth powerdomain as ideal completion

$X, Y \subseteq \mathsf{K}(D)$. $X \sqsubseteq^\sharp Y$ iff $(\forall y \in Y)(\exists x \in X)(x \sqsubseteq y)$

Smyth powerdomain of a coherent algebraic cpo: proper ideal completion of the set of all finite subsets of D, ordered by $\sqsubseteq^\sharp$.

Used for modelling nondeterminism in domain theory.

In the following: $(D, \sqsubseteq)$ coherent algebraic domain.

Logic RZ

(Rounds & Zhang, 2001)

clause X: finite subset of $\mathsf{K}(D)$

$w \in D$: $w \models X$ iff $(\exists x \in X)(x \sqsubseteq w)$.

theory T: set of clauses.

$w \models T$ iff $(\forall X \in T)(w \models X)$.

$T \models X$ iff $(\forall w \in D)(w \models T \implies w \models X)$.

Logic RZ

Proof theory: (WLP'02)

$$\overline{\{\bot\}}$$

$$\frac{X; \qquad a \in X; \qquad y \sqsubseteq a}{\{y\} \cup (X \setminus \{a\})}$$

$$\frac{X; \qquad y \in \mathsf{K}(D)}{\{y\} \cup X}$$

$$\frac{X_1 \quad X_2; \qquad a_1 \in X_1 \quad a_2 \in X_2}{\mathrm{mub}\{a_1, a_2\} \cup (X_1 \setminus \{a_1\}) \cup (X_2 \setminus \{a_2\})}$$

Logic RZ is compact.

Smyth powerdomain via the logic RZ

(Rounds & Zhang 2001)

The logically closed theories are the ideals under $\sqsubseteq^\sharp$.

Smyth powerdomain: consistent closed theories under set-inclusion.

Relation to Formal Concept Analysis (FCA)

(FCA: tool used in data mining and analysis; Ganter & Wille 1999)

G set of objects; M set of attributes. $C \subseteq G \times M$ *formal context.*

$A \subseteq G$ then $A' = \{m \in M \mid (\forall g \in A)(g, m) \in C\}$.
$B \subseteq M$ then $B' = \{g \in G \mid (\forall m \in B)(g, m) \in C\}$.

Formal concept: Pair (A, B) with $A' = B$, $A = B'$.
Equivalently: All (B', B'') for $B \subseteq M$.

Formal concept lattice:
complete lattice of all concepts ordered by $\supseteq$ in second argument.

Relation to Formal Concept Analysis (FCA)

(with Matthias Wendt, ICCS 2003)

Consider subposet D of all $(\{b\}', \{b\}'')$, $b \in M$,
and all $(\{a\}'', \{a\}')$, $a \in G$, ordered reversely (add $\perp$).

If D is a coherent algebraic cpo (eg. all finite cases), then

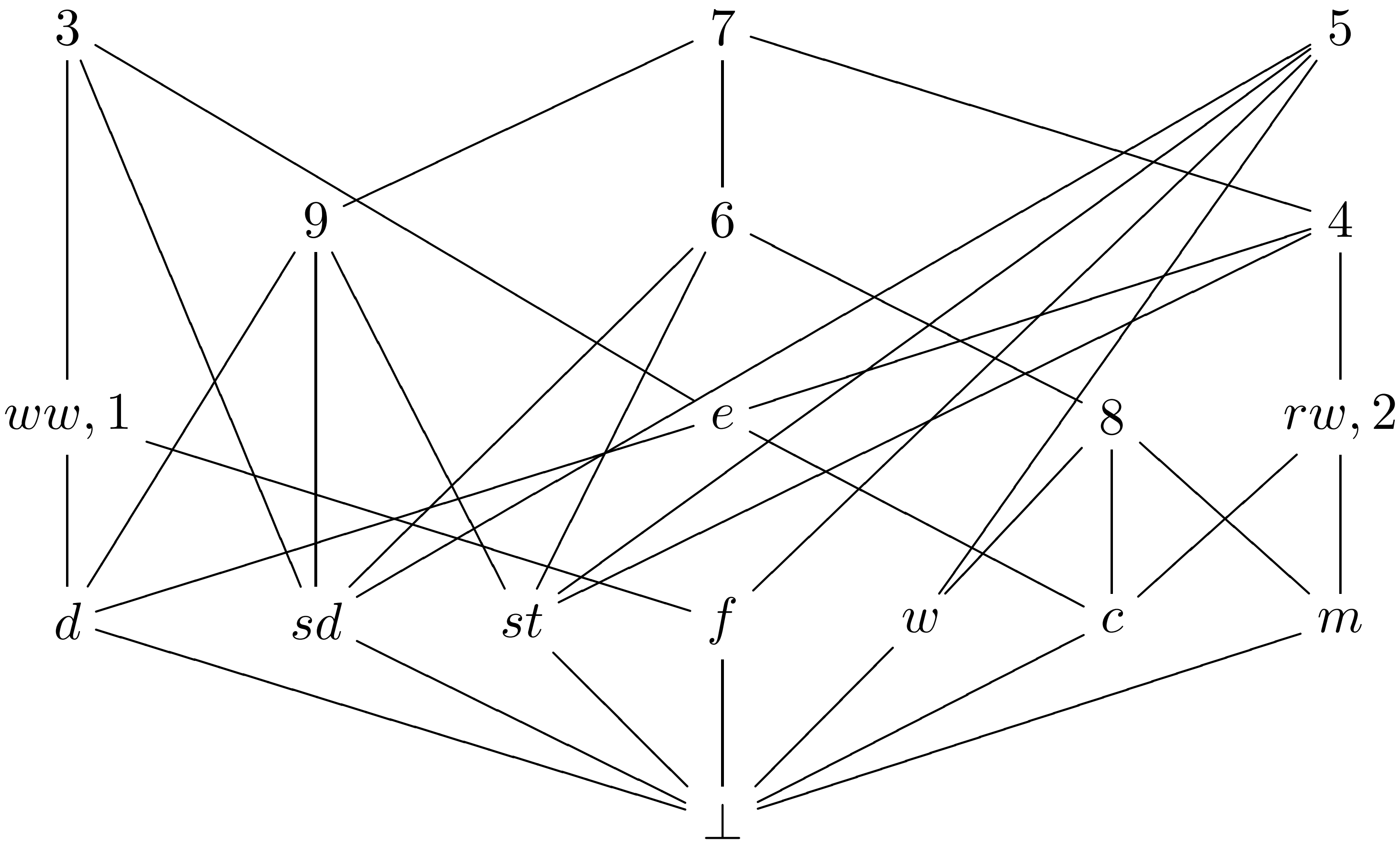
for $\mathsf{K}(D) \supseteq \{b_i \mid i \in I\} = B \subseteq M$ we have

$B'' = \{b \in M \mid \{\{b_i\} \mid i \in I\} \models \{b\}\}$.

Relation to Formal Concept Analysis (FCA)

	salad	starter	fish	meat	red wine	white wine	water	dessert	coffee	expensive
1			×			×		×		
2				×	×			×		
3	×		×			×		×	×	×
4		×		×	×			×	×	×
5	×	×	×				×			
6	×	×		×			×		×	
7	×	×		×	×		×	×	×	×
8				×			×		×	
9	×	×						×		

Relation to Formal Concept Analysis (FCA)



Logic programming in coherent algebraic domains

(Rounds & Zhang 2001)

Add material implication: $X \leftarrow Y$ for X, Y clauses.

$w \models P$: if $w \models Y$ for $X \leftarrow Y \in P$, then $w \models X$.

Propagation rule CP(P):

$$\frac{X_1 \quad \dots \quad X_n; \quad a_i \in X_i; \quad Y \leftarrow Z \in P; \quad \mathrm{mub}\{a_1, \dots, a_n\} \models Z}{Y \cup \bigcup_{i=1}^{n}(X_i \setminus \{a_i\})}$$

Semantic operator on theories:

$$\mathcal{T}_P(T) = \mathrm{cons}\left(\{Y \mid Y \text{ is a } \mathrm{CP}(P)\text{-consequence of } T\}\right).$$

- ▶ $\mathcal{T}_P$ is Scott continuous [RZ01].
- ▶ $\mathrm{fix}\left(\mathcal{T}_P\right) = \mathrm{cons}(P)$.

Additon of default negation

Extended rules: $X \leftarrow Y, {\sim}Z$.

P program, T theory. Define P/T:

Replace $Y, {\sim}Z$ by Y if $T \not\models Z$.

Remove rule if $T \models Z$.

T *answer theory* for P if $T = \mathrm{cons}(P/T) = \mathrm{fix}(\mathcal{T}_{P/T})$.

A version of default logic

Consider $\mathbb{T}^\omega$.

Clauses are the propositional formulae in disjunctive normal form.

Extended rules correspond to defaults.

Answer theories correspond to default extensions.

But logical consequence is not classical.

▶ On $\mathbb{T}^\omega$ we obtain something akin to propositional default logic.

Answer set programming

We do the same with *models*. (?!)

P program, $w \in D$. Define P/w:

Replace $Y, \sim Z$ by Y if $w \not\models Z$.

Remove rule if $w \models Z$.

w *min-answer model* for P if w is minimal with $w \models \mathrm{fix}(\mathcal{T}_{P/w})$.

Answer set programming

Consider $\mathbb{T}^\omega$.

Consider programs P with rules $X \leftarrow Y, \sim Z$ such that:

Y singleton clause

X, Y, Z contain only atoms in $\mathbb{T}^\omega$ or $\perp$

These programs are exactly extended disjunctive programs.

Min-answer models w correspond to *answer sets* $\{L \text{ atom} \mid w \models \{L\}\}$.

Further Work

What *is* this version of default logic?

Syntactic extensions/integrating paradigms

(FCA) applications

Decidability issues