de 1

`flies`$(x) \leftarrow$ `bird`$(x) \land \neg$`penguin`(x)

`bird`(Bob) $\leftarrow$

Does Bob fly?

Fixed-point semantics in logic programming and nonmonotonic reasoning: A uniform approach

Pascal Hitzler

Workshop on Proof Theory and Computation, Dresden, May 2002

de 2

Contents

For semantics based on monotonic operators:
characterizations using level mappings.

For semantics based on non-monotonic operators:
study of dynamic behaviour using level mappings.

Artificial Intelligence Institute, Dresden University of Technology, Dresden, Germany

Slide 3

Restrictions

We work on ground instantiations of normal logic programs.

Negation symbols may appear in clause bodies.

Essentially, program P is countably infinite set of propositional rules.
Herbrand base $B_P \approx$ set of propositional variables (atoms).

$$A \leftarrow B_1, \ldots, B_n, \neg C_1, \ldots, \neg C_m$$
$$A \leftarrow \texttt{body}$$

(We talk about Prolog only in a very abstract sense.)

Slide 4

Level mappings

Level mapping $B_P \to \alpha$ for ordinal α.
ω -level mapping $B_P \to \omega$.

- order on atoms
- precedence
- dependence
- distance

Least models

Slide 5

Positive (definite) program P.

There is a unique model M of P for which there exists a level mapping $l : B_P \to \alpha$ such that for each $A \in B_P$ with $M \models A$ there exists $A \leftarrow \texttt{body}$ in P with $M \models \texttt{body}$ and $l(A) > l(B)$ for each $B \in \texttt{body}$.

$M = T_P{\uparrow}\omega = \text{lfp}(T_P)$ is the least model of P.

$l(A) = \min\{n \mid A \in T_P{\uparrow}(n+1)\}$.

Stable models

Slide 6

(Fages 1994)

P normal (with negation).

A model M of P is stable if and only if there exists a level mapping $l : B_P \to \alpha$ such that for each $A \in B_P$ with $M \models A$ there exists $A \leftarrow body$ in P with $M \models \texttt{body}$ and $l(A) > l(B)$ for all $B \in \texttt{body}^+$.

$\texttt{body}^+$: all atoms occuring positively in $\texttt{body}$.

$M = \text{GL}_P(M) = T_{P/M}{\uparrow}\omega = \text{lfp}(T_{P/M})$.

$l(A) = \min\{n \mid A \in T_{P/M}{\uparrow}(n+1)\}$.

Kleene's strong three-valued logic

Slide 7

Thruth values $f < u < t$, $\wedge = \min$, $\vee = \max$, $\neg$ as expected.

Interpretations: consistent signed sets of atoms.
$I = I^+ \dot{\cup} \neg I^- \subseteq B_P \cup \neg B_P$.

I^+: true atoms
I^-: false atoms

Signed: contains atoms and negated atoms.
Consistent: does not contain both A and $\neg A$.

With order $I \subseteq K$: Plotkin's domain $\mathbb{T}^\omega$.

I-partial level mapping:
partial mapping $l : B_P \to \alpha$ with $\text{dom}(l) = I^+ \cup I^-$.
Set $l(\neg A) = l(A)$.

Fitting models

Slide 8

There is a greatest model M of P such that there is an M-partial level mapping l for P such that each $A \in \text{dom}(l)$ satisfies one of the following conditions.

(Fi) $A \in M$ and there exists $A \leftarrow L_1, \ldots, L_n$ in P such that for all i we have $L_i \in M$ and $l(A) > l(L_i)$.

(Fii) $\neg A \in M$ and for each $A \leftarrow L_1, \ldots, L_n$ in P there exists i with $\neg L_i \in M$ and $l(A) > l(L_i)$.

$M = \Phi_P{\uparrow}\alpha = \text{lfp}(\Phi_P)$ Fitting model.

$l(A) = \min\{\beta \mid A \in \Phi_P{\uparrow}(\beta+1)\}$.

Well-founded models

(Hitzler & Wendt 2002)

Replace

(Fii) $\neg A \in M$ and for $A \leftarrow L_1, \ldots, L_n$ in P there exists i with $\neg L_i \in M$ and $l(A) > l(L_i)$.

by

(WFii) $\neg A \in M$ and for each $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in P one of the following holds:

- (WFiia) There exists i with $\neg A_i \in M$ and $l(A) \geq l(A_i)$.
- (WFiib) There exists j with $B_j \in M$ and $l(A) > l(B_j)$.

Prevent recursion through negation: Idea behind local stratification.

Weak stratification: Presentation by Matthias Wendt next Wednesday.

Slide 9

Well-founded models

There is a greatest model M of P such that there is an M-partial level mapping l for P such that each $A \in \mathrm{dom}(l)$ satisfies one of the following conditions.

(Fi) $A \in M$ and there exists $A \leftarrow L_1, \ldots, L_n$ in P such that for all i we have $L_i \in M$ and $l(A) > l(L_i)$.

(WFii) $\neg A \in M$ and for each $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in P one of the following holds:

- (WFiia) There exists i with $\neg A_i \in M$ and $l(A) \geq l(A_i)$.
- (WFiib) There exists j with $B_j \in M$ and $l(A) > l(B_j)$.

$M = W_P{\uparrow}\alpha = \mathrm{lfp}(W_P)$ well-founded model.

$l(A) = \min\{\beta \mid A \in W_P{\uparrow}(\beta + 1)\}$.

Slide 10

Well-founded models

stable models: $M = \mathrm{GL}_P(M) = T_{P/M}{\uparrow}\omega$.

GL_P antitonic, GL_P^2 monotonic.

well-founded model:

$$\mathrm{lfp}\,(\mathrm{GL}_P^2) \cup \neg\mathrm{gfp}\,(\mathrm{GL}_P^2) = \mathrm{lfp}\,(\mathrm{GL}_P^2) \cup \neg\mathrm{GL}_P\,(\mathrm{lfp}\,(\mathrm{GL}_P^2)).$$

$L_\alpha = \mathrm{GL}_P^2{\uparrow}\alpha \qquad G_\alpha = \mathrm{GL}_P(L_\alpha)$.

$l(A) = (\alpha, n)$ with:

For $A \in \mathrm{lfp}\,(\mathrm{GL}_P^2)$: α least with $A \in L_{\alpha+1}$, n least with $A \in T_{P/G_\alpha}{\uparrow}(n+1)$.

For $A \notin \mathrm{gfp}\,(\mathrm{GL}_P^2)$: α least with $A \notin G_{\beta+1}$, $n = \omega$.

Slide 11

Supported models

Back to classical (two-valued) logic.

Immediate consequence operator:
$T_P(I)$ set of all $A \in B_P$ such that exists $A \leftarrow$ `body` in P with $I \models$ `body`.

T_P in general not monotonic.

supported model: $M = T_P(M)$.

propagation along $\leftarrow$

Slide 12

le 13

Related paradigms

logic programs with immediate consequence operator

cellular automata

artificial neural networks

topological dynamical systems

(see e.g. Blair et al. 1999)

le 14

Acyclic/locally hierarchical programs

P *locally hierarchical* if $l : B_P \to \alpha$ for some ordinal α
and for each $A \leftarrow L_1, \dots, L_n$ in P:

$$l(A) > l(L_i) \text{ for all } i.$$

P *acyclic* if $l : B_P \to \omega$.

Distance function on space I_P of all interpretations:

$$d(J,K) = \begin{cases} \inf \left\{2^{-\beta} \mid J, K \text{ agree on atoms with level } < \beta\right\} & \text{if } J \neq K \\ 0 & \text{if } J = K. \end{cases}$$

Slide 15

Acyclic programs

P acyclic:

- d complete ultrametric.
- T_P contraction.
- T_P has unique fixed point. (Via Banach contraction mapping theorem.)
- P has unique supported model M.
- $T_P^n(K) \to M$ in the Cantor topology on I_P (for all K).

Acyclic programs terminate under SLDNF-resolution
with respect to any selection rule. (Bezem 1989)

Slide 16

Locally hierarchical programs

P locally hierarchical:

- d spherically complete generalized ultrametric. (d maps into poset.)
- T_P strictly contracting.
- T_P has unique fixed point. (Via Priess-Crampe & Ribenboim theorem.)
- P has unique supported model M.
- $T_P^\alpha(K) = M$ for some α via transfinite iteration.

Acceptable programs

Neg_P^*: atoms occuring negatively in P
together with all predicates on which they depend.
P^-: all ground clauses with head in Neg_P^*.

le 17

P is *acceptable* (with respect to model I and $l : B_P \to \omega$) if
I restricted to Neg_P^* is a supported model of P^-
and for all $A \leftarrow L_1, \ldots, L_n$ in P and all $i \in \{1, \ldots, n\}$:
if $I \models \bigwedge_{j=1}^{i-1} L_i$ then $l(A) > l(L_i)$.

Acceptable programs are left-terminating (and conversely if non-floundering) (Apt & Pedreschi 1994).

Acceptable programs

For $K \in I_P$ let K' be K restricted to predicates not in Neg_P^*.

$$f : I_P \to \mathbb{N} : K \mapsto \begin{cases} 0 & \text{if } K \subseteq I \\ 2^{-n} & n \text{ least s.t. exists } A \in K \setminus I \text{ with } l(A) = n. \end{cases}$$

$u(K) = \max\{f(K'), d(K \setminus K', I \setminus I')\}$

le 18

$\varrho(J, K) = \max\{d(J, K), u(J), u(K)\}$.

ϱ almost a metric, fails $d(K, K) = 0$ for all K.
ϱ dislocated metric (Matthews 1986).

Banach theorem carries over (Matthews 1986) with same consequences as for acyclic programs.

Used a priori (partial) knowledge about fixed point.

Full knowledge about fixed point

(Hitzler & Seda 2001)

Let (X, τ) be a T_1 topological space and $f : X \to X$ be a function which has a unique fixed point a and such that for each $x \in X$ we have that $f^n(x)$ converges to a in τ.

Slide 19

Then there exists a function $d : X \times X \to \mathbb{R}$ such that (X, d) is a complete ultrametric space and such that for all $x, y \in X$ we have $d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$.

In the proof: d is constructed using a.

"Converse" of the Banach contraction mapping theorem.

Quo Vadis? Fixed-point theorems

Tools have been developed.

Mostly new perspectives on known results.

Slide 20

Are there new applications out there?

Acknowledgements
Anthony K. Seda, Cork.
Most of the fixed-point results were joint work.
Matthias Wendt.
Main work in the "well-founded" characterization.

Quo Vadis? Characterizations via level-mappings

Generalize

- extended disjunctive etc. programs.
- logic programming on posets. (Rounds & Zhang 2001; Hitzler 2002)

Apply

- computation of models (answer set programming).

Acknowledgements

Anthony K. Seda, Cork.

Most of the fixed-point results were joint work.

Matthias Wendt.

Main work on the “well-founded” characterization.